Factoring - Factoring Special Products

Objective: Identify and factor special products including a difference of squares, perfect squares, and sum and difference of cubes.

When factoring there are a few special products that, if we can recognize them, can help us factor polynomials. The first is one we have seen before. When multiplying special products we found that a sum and a difference could multiply to a difference of squares. Here we will use this special product to help us factor

$$\textbf{Difference of Squares: } \boldsymbol{a^2 - b^2 = (a+b)(a-b)}$$

If we are subtracting two perfect squares then it will always factor to the sum and difference of the square roots.

Example 1.

$x^2 - 16$	Subtracting two perfect squares, the square roots are x and 4
$(x+4)(x-4)$	Our Solution

Example 2.

$9a^2 - 25b^2$	Subtracting two perfect squares, the square roots are $3a$ and $5b$
$(3a+5b)(3a-5b)$	Our Solution

It is important to note, that a sum of squares will never factor. It is always prime. This can be seen if we try to use the ac method to factor $x^2 + 36$.

Example 3.

$x^2 + 36$	No bx term, we use $0x$.
$x^2 + 0x + 36$	Multiply to 36, add to 0
$1\cdot 36, 2\cdot 18, 3\cdot 12, 4\cdot 9, 6\cdot 6$	No combinations that multiply to 36 add to 0
Prime, cannot factor	Our Solution

It turns out that a sum of squares is always prime.

$$\textbf{Sum of Squares: } \boldsymbol{a^2 + b^2} = \textbf{Prime}$$

A great example where we see a sum of squares comes from factoring a difference of 4th powers. Because the square root of a fourth power is a square ($\sqrt{a^4} = a^2$), we can factor a difference of fourth powers just like we factor a difference of squares, to a sum and difference of the square roots. This will give us two factors, one which will be a prime sum of squares, and a second which will be a difference of squares which we can factor again. This is shown in the following examples.

Example 4.

$a^4 - b^4$	Difference of squares with roots a^2 and b^2
$(a^2 + b^2)(a^2 - b^2)$	The first factor is prime, the second is a difference of squares!
$(a^2 + b^2)(a + b)(a - b)$	Our Solution

Example 5.

$x^4 - 16$	Difference of squares with roots x^2 and 4
$(x^2 + 4)(x^2 - 4)$	The first factor is prime, the second is a difference of squares!
$(x^2 + 4)(x + 2)(x - 2)$	Our Solution

Another factoring shortcut is the perfect square. We had a shortcut for multiplying a perfect square which can be reversed to help us factor a perfect square

$$\textbf{Perfect Square: } \boldsymbol{a^2 + 2\mathbf{ab} + b^2 = (a + b)^2}$$

A perfect square can be difficult to recognize at first glance, but if we use the ac method and get two of the same numbers we know we have a perfect square. Then we can just factor using the square roots of the first and last terms and the sign from the middle. This is shown in the following examples.

Example 6.

$x^2 - 6x + 9$	Multiply to 9, add to -6
	The numbers are -3 and -3, the same! Perfect square
$(x - 3)^2$	Use square roots from first and last terms and sign from the middle

Example 7.

$4x^2 + 20xy + 25y^2$	Multiply to 100, add to 20
	The numbers are 10 and 10, the same! Perfect square
$(2x + 5y)^2$	Use square roots from first and last terms and sign from the middle

World View Note: The first known record of work with polynomials comes from the Chinese around 200 BC. Problems would be written as "three sheafs of a good crop, two sheafs of a mediocre crop, and one sheaf of a bad crop sold for 29 dou. This would be the polynomial (trinomial) $3x + 2y + z = 29$.

Another factoring shortcut has cubes. With cubes we can either do a sum or a difference of cubes. Both sum and difference of cubes have very similar factoring formulas

$$\textbf{Sum of Cubes: } \boldsymbol{a^3 + b^3 = (a + b)(a^2 - \mathrm{ab} + b^2)}$$

$$\textbf{Difference of Cubes: } \boldsymbol{a^3 - b^3 = (a - b)(a^2 + \mathrm{ab} + b^2)}$$

Comparing the formulas you may notice that the only difference is the signs in between the terms. One way to keep these two formulas straight is to think of SOAP. S stands for Same sign as the problem. If we have a sum of cubes, we add first, a difference of cubes we subtract first. O stands for Opposite sign. If we have a sum, then subtraction is the second sign, a difference would have addition for the second sign. Finally, AP stands for Always Positive. Both formulas end with addition. The following examples show factoring with cubes.

Example 8.

$$\begin{array}{rl} m^3 - 27 & \text{We have cube roots } m \text{ and } 3 \\ (m \quad 3)(m^2 \quad 3m \quad 9) & \text{Use formula, use SOAP to fill in signs} \\ (m - 3)(m^2 + 3m + 9) & \text{Our Solution} \end{array}$$

Example 9.

$$\begin{array}{rl} 125p^3 + 8r^3 & \text{We have cube roots } 5p \text{ and } 2r \\ (5p \quad 2r)(25p^2 \quad 10r \quad 4r^2) & \text{Use formula, use SOAP to fill in signs} \\ (5p + 2r)(25p^2 - 10r + 4r^2) & \text{Our Solution} \end{array}$$

The previous example illustrates an important point. When we fill in the trinomial's first and last terms we square the cube roots $5p$ and $2r$. Often students forget to square the number in addition to the variable. Notice that when done correctly, both get cubed.

Often after factoring a sum or difference of cubes, students want to factor the second factor, the trinomial further. As a general rule, this factor will always be prime (unless there is a GCF which should have been factored out before using cubes rule).

The following table sumarizes all of the shortcuts that we can use to factor special products

Factoring Special Products

Difference of Squares	$a^2 - b^2 = (a+b)(a-b)$
Sum of Squares	$a^2 + b^2 = \text{Prime}$
Perfect Square	$a^2 + 2ab + b^2 = (a+b)^2$
Sum of Cubes	$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$
Difference of Cubes	$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

As always, when factoring special products it is important to check for a GCF first. Only after checking for a GCF should we be using the special products. This is shown in the following examples

Example 10.

$72x^2 - 2$	GCF is 2
$2(36x^2 - 1)$	Difference of Squares, square roots are $6x$ and 1
$2(6x+1)(6x-1)$	Our Solution

Example 11.

$48x^2y - 24xy + 3y$	GCF is $3y$
$3y(16x^2 - 8x + 1)$	Multiply to 16 add to 8
	The numbers are 4 and 4, the same! Perfect Square
$3y(4x-1)^2$	Our Solution

Example 12.

$128a^4b^2 + 54ab^5$	GCF is $2ab^2$
$2ab^2(64a^3 + 27b^3)$	Sum of cubes! Cube roots are $4a$ and $3b$
$2\text{ab}^2(4a+3b)(16a^2 - 12ab + 9b^2)$	Our Solution

6.5 Practice - Factoring Special Products

Factor each completely.

1) r^2-16

2) x^2-9

3) v^2-25

4) x^2-1

5) p^2-4

6) $4v^2-1$

7) $9k^2-4$

8) $9a^2-1$

9) $3x^2-27$

10) $5n^2-20$

11) $16x^2-36$

12) $125x^2+45y^2$

13) $18a^2-50b^2$

14) $4m^2+64n^2$

15) a^2-2a+1

16) k^2+4k+4

17) x^2+6x+9

18) $n^2-8n+16$

19) x^2-6x+9

20) k^2-4k+4

21) $25p^2-10p+1$

22) x^2+2x+1

23) $25a^2+30ab+9b^2$

24) $x^2+8xy+16y^2$

25) $4a^2-20ab+25b^2$

26) $18m^2-24mn+8n^2$

27) $8x^2-24xy+18y^2$

28) $20x^2+20xy+5y^2$

29) $8-m^3$

30) x^3+64

31) x^3-64

32) x^3+8

33) $216-u^3$

34) $125x^3-216$

35) $125a^3-64$

36) $64x^3-27$

37) $64x^3+27y^3$

38) $32m^3-108n^3$

39) $54x^3+250y^3$

40) $375m^3+648n^3$

41) a^4-81

42) x^4-256

43) $16-z^4$

44) n^4-1

45) x^4-y^4

46) $16a^4-b^4$

47) m^4-81b^4

48) $81c^4-16d^4$

6.5

Answers - Factoring Special Products

1) $(r+4)(r-4)$
2) $(x+3)(x-3)$
3) $(v+5)(v-5)$
4) $(x+1)(x-1)$
5) $(p+2)(p-2)$
6) $(2v+1)(2v-1)$
7) $(3k+2)(3k-2)$
8) $(3a+1)(3a-1)$
9) $3(x+3)(x-3)$
10) $5(n+2)(n-2)$
11) $4(2x+3)(2x-3)$
12) $5(25x^2+9y^2)$
13) $2(3a+5b)(3a-5b)$
14) $4(m^2+16n^2)$
15) $(a-1)^2$
16) $(k+2)^2$
17) $(x+3)^2$
18) $(n-4)^2$
19) $(x-3)^2$
20) $(k-2)^2$
21) $(5p-1)^2$
22) $(x+1)^2$
23) $(5a+3b)^2$
24) $(x+4y)^2$
25) $(2a-5b)^2$
26) $2(3m-2n)^2$
27) $2(2x-3y)^2$
28) $5(2x+y)^2$
29) $(2-m)(4+2m+m^2)$
30) $(x+4)(x^2-4x+16)$
31) $(x-4)(x^2+4x+16)$
32) $(x+2)(x^2-2x+4)$
33) $(6-u)(36+6u+u^2)$
34) $(5x-6)(25x^2+30x+36)$
35) $(5a-4)(25a^2+20a+16)$
36) $(4x-3)(16x^2+12x+9)$
37) $(4x+3y)(16x^2-12xy+9y^2)$
38) $4(2m-3n)(4m^2+6mn+9n^2)$
39) $2(3x+5y)(9x^2-15xy+25y^2)$
40) $3(5m+6n)(25m^2-30mn+36n^2)$
41) $(a^2+9)(a+3)(a-3)$
42) $(x^2+16)(x+4)(x-4)$
43) $(4+z^2)(2+z)(2-z)$
44) $(n^2+1)(n+1)(n-1)$
45) $(x^2+y^2)(x+y)(x-y)$
46) $(4a^2+b^2)(2a+b)(2a-b)$
47) $(m^2+9b^2)(m+3b)(m-3b)$
48) $(9c^2+4d^2)(3c+2d)(3c-2d)$